

# Rational solutions to the ABS list: Transformation approach

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February 14, 2017

## Abstract

In the paper we derive rational solutions for the lattice potential modified Korteweg-de Vries equation, and Q2, Q1( $\delta$ ), H3( $\delta$ ), H2 and H1 in the Adler-Bobenko-Suris list. Bäcklund transformations between these lattice equations are used. All these rational solutions are related to a unified  $\tau$  function in Casoratian form which obeys a bilinear superposition formula.

**Keywords:** rational solutions, Bäcklund transformation, Casoratian, ABS list

**PACS numbers:** 02.30.Ik, 05.45.Yv

## 1 Introduction

In recent decades the research of discrete integrable systems has undergone rapid progress (see [1] and the references therein). As a new concept, multidimensional consistency, allowing suitable lattice equations to be embedded into a higher dimensional space in a consistent way, has played an important role in the research of quadrilateral equations [2–7]. Quadrilateral equations that are consistent around the cube (CAC) were searched and classified by Adler, Bobenko and Suris (ABS) [5] and in their list only 9 equations are included: Q4, Q3( $\delta$ ), Q2, Q1( $\delta$ ), A2, A1( $\delta$ ), H3( $\delta$ ), H2 and H1. All these equations have been solved from different approaches [8–14].

As for rational solutions, which are solutions expressed by fractions of polynomials, in general, such type of solutions can be derived from soliton solutions through a special limit procedure (or a Taylor expansion), which corresponds to a way to generate multiple zero eigenvalues for certain spectral problems (see [15, 16] as examples). For the  $\delta$ -dependent equations in the ABS list, for example, H3( $\delta$ ) and Q1( $\delta$ ), the existence of  $\delta$  (i.e.  $\delta \neq 0$ ) plays a crucial role [14] in the procedure of obtaining rational solutions from their soliton solutions. For H1 which is independent of  $\delta$ , its rational solutions were obtained recently by making use of the Hirota-Miwa equation and a continuous auxiliary variable [17]. Besides, as a generic (2+1)-D bilinear model, polynomial solutions of the Hirota-Miwa equation have been derived from several ways and presented via different forms [18, 19].

In this paper we systematically construct rational solutions for the ABS list using Bäcklund transformations (BTs). A fundamental role playing in the paper is the lattice potential modified Korteweg-de Vries (lpmKdV) equation. There is a non-auto BT which connects the lpmKdV equation and Q1(0) (also known as the lattice Schwarzian Korteweg-de Vries equation and

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cross-ratio equation). The two equations and their BT constitute a consistent triplet, say, viewing the BT as a two-component system, then the compatibility of each component yields a lattice equation of another component which is in the triplet. This means any pair of solutions of the BT provide solutions to the two equations that the BT connects. Details will be shown in Sec.3.1 and 3.2 on how such a consistent triplet works in generating rational solutions. We also make use of non-auto BTs between equations in the ABS list [20]. Starting from the lpmKdV equation and Q1(0), rational solutions of Q2, Q1( $\delta$ ), A1( $\delta$ ), H3( $\delta$ ), H2 and H1 in the ABS list can be derived following the map:

$$\begin{array}{ccccccc}
\text{H2} & & \text{Q1(0)} & \longrightarrow & \text{Q1}(\delta) & \longrightarrow & \text{Q2} \\
\updownarrow & & \updownarrow & & \downarrow & & \\
\text{H1} & \longleftarrow & \text{lpmKdV} & & \text{A1}(\delta) & \longleftrightarrow & \text{H3}(\delta)
\end{array}$$

Fig.1 A map for generating rational solutions.

In the map double-head arrow means the two equations it connects and their BT form a consistent triplet.

Moreover, we find all the obtained rational solutions are related to a unified  $\tau$  function in Casoratian form which obeys a bilinear superposition formula (see (65)). Compared with those rational solutions of H3( $\delta$ ) and Q1( $\delta$ ) derived in [14], here we obtain new solutions. In fact, we will see that rational solutions of Q1( $\delta$ ) can explicitly be expressed through the rational solutions of Q1(0). Similar results hold for H3( $\delta$ ) as well.

The paper is organized as follows. In Sec.2 as preliminary we list quadrilateral equations that we consider in the paper and some notations. Then in Sec.3 and Sec.4 we derive some rational solutions for the equations listed in Sec.2. In Sec.5 rational solutions in Casoratian form are proved. Finally in Sec.6 we give conclusions.

## 2 Preliminary

We list quadrilateral equations that we consider in the paper.

$$\text{H1 : } (\tilde{u} - \hat{u})(\hat{\tilde{u}} - u) = q - p, \quad (1)$$

$$\text{H2 : } (\tilde{v} - \hat{v})(v - \hat{\tilde{v}}) + (q - p)(v + \tilde{v} + \hat{v} + \hat{\tilde{v}}) + q^2 - p^2 = 0, \quad (2)$$

$$\text{lpmKdV : } a(V\tilde{V} - \hat{V}\hat{\tilde{V}}) - b(V\hat{V} - \tilde{V}\hat{\tilde{V}}) = 0, \quad (3)$$

$$\text{H3 : } a(Z\tilde{Z} + \hat{Z}\hat{\tilde{Z}}) - b(Z\hat{Z} + \tilde{Z}\hat{\tilde{Z}}) + 2\delta(a^2 - b^2) = 0, \quad (4)$$

$$\text{Q1(0) : } p(v - \hat{v})(\tilde{v} - \hat{\tilde{v}}) - q(v - \tilde{v})(\hat{v} - \hat{\tilde{v}}) = 0, \quad (5)$$

$$\text{Q1}(\delta) : p(u - \hat{u})(\tilde{u} - \hat{\tilde{u}}) - q(u - \tilde{u})(\hat{u} - \hat{\tilde{u}}) + \delta^2 pq(p - q) = 0, \quad (6)$$

$$\text{A1}(\delta) : p(z + \hat{z})(\tilde{z} + \hat{\tilde{z}}) - q(z + \tilde{z})(\hat{z} + \hat{\tilde{z}}) - \delta^2 pq(p - q) = 0, \quad (7)$$

$$\begin{aligned} \text{Q2 : } & p(w - \hat{w})(\tilde{w} - \hat{\tilde{w}}) - q(w - \tilde{w})(\hat{w} - \hat{\tilde{w}}) + pq(p - q)(w + \tilde{w} + \hat{w} + \hat{\tilde{w}}) \\ & - pq(p - q)(p^2 - pq + q^2) = 0. \end{aligned} \quad (8)$$

Here we use conventional notations  $\tilde{u} \doteq u_{n+1,m}$ ,  $\hat{u} \doteq u_{n,m+1}$ . In the above equations,  $p$  and  $a$  are spacing parameters of  $n$ -direction and  $q$  and  $b$  are of  $m$ -direction;  $\delta$  is an arbitrary constant.

Casoratian is a discrete version of Wronskian. Suppose that a basic column vector is

$$\psi(n, m, l) = (\psi_1(n, m, l), \psi_2(n, m, l), \dots, \psi_N(n, m, l))^T. \quad (9)$$

Introduce an operator  $E_\nu$  to denote

$$E_n^s \psi \equiv \psi(n+s, m, l), \quad E_m^s \psi \equiv \psi(n, m+s, l), \quad E_l^s \psi \equiv \psi(n, m, l+s).$$

Then a  $N$ th-order Casoratian w.r.t.  $E_l$ -shift is defined by

$$|\psi, E_l \psi, E_l^2 \psi, E_l^3 \psi, \dots, E_l^{N-1} \psi|,$$

and usually is compactly written as (cf. [21])

$$|\widehat{N-1}| = |0, 1, 2, \dots, N-1|.$$

Another notation which is often used is  $|\widehat{N-2}, N| = |0, 1, \dots, N-2, N|$ .

Besides, in Wronskian/Casoratian verification of solutions to a bilinear equation, the equation is usually reduced to a Laplace expansion of a zero-valued  $2N \times 2N$  determinant. The expansion is described as:

**Lemma 1.** [21] Suppose that  $\mathbf{B}$  is an  $N \times (N-2)$  matrix and  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$  are  $N$ th-order column vectors, then

$$|\mathbf{B}, \mathbf{a}, \mathbf{b}| |\mathbf{B}, \mathbf{c}, \mathbf{d}| - |\mathbf{B}, \mathbf{a}, \mathbf{c}| |\mathbf{B}, \mathbf{b}, \mathbf{d}| + |\mathbf{B}, \mathbf{a}, \mathbf{d}| |\mathbf{B}, \mathbf{b}, \mathbf{c}| = 0. \quad (10)$$

### 3 Rational solutions to lpmKdV, Q1, H3 and Q2

In this section we first investigate relation between the lpmKdV equation and Q1(0). Such a relation will be used to construct rational solutions to not only the two equations themselves but also to Q1( $\delta$ ), H3( $\delta$ ) and Q2.

#### 3.1 Solution sequence of Q1(0) and lpmKdV

Q1(0) is the equation (6) with  $\delta = 0$ , i.e.

$$\text{Q1}(0) : p(v - \widehat{v})(\widetilde{v} - \widehat{\widehat{v}}) - q(v - \widetilde{v})(\widehat{v} - \widehat{\widehat{v}}) = 0. \quad (11)$$

Between it and the lpmKdV equation (3) there is a non-auto Bäcklund transformation [10]

$$\widetilde{v} - v = aV\widetilde{V}, \quad (12a)$$

$$\widehat{v} - v = bV\widehat{V}, \quad (12b)$$

where

$$p = a^2, \quad q = b^2. \quad (13)$$

Equations (11), (3) and (12) constitute a consistent triplet in the following sense: as an equation set, the compatibility of  $\widetilde{v} = \widehat{\widehat{v}}$  and  $\widetilde{V} = \widehat{\widehat{V}}$  respectively yield (3) and (11).

Such an consistency can be used to construct solutions for equation (11) and (3):

**Lemma 2.** With the consistent triplet (11), (3) and (12), we have the following:

(1). Starting from any solution  $v$  of (11), by integration through (12), the resulted  $V$  solves equation (3), and vice versa; (2). any solution pair  $(v, V)$  of (12) gives a solution  $v$  to (11) and  $V$  to (3).

Further than that, we have

**Lemma 3.** *For an arbitrary solution pair  $(v, V)$  of (12) where  $V \neq 0$ , function  $V_1 = v/V$  solves the lpmKdV equation (3).*

*Proof.* Substituting  $V_1 = v/V$  into (3) and making use of relation (12), it is easy to check  $V_1$  satisfies (3).  $\square$

This lemma provides an approach to generate a sequence of solution pairs of the BT (12).

**Theorem 1.** *For any solution pair  $(v_N, V_N)$  of the BT (12), define*

$$V_{N+1} = \frac{v_N}{V_N}, \quad (14a)$$

*and it solves the lpmKdV equation (3). Next, through the BT (12), i.e.*

$$\tilde{v}_{N+1} - v_{N+1} = aV_{N+1}\tilde{V}_{N+1}, \quad \hat{v}_{N+1} - v_{N+1} = bV_{N+1}\hat{V}_{N+1}, \quad (14b)$$

*function  $v_{N+1}$  is well defined and it solves Q1(0) (11).  $v_N$  and  $v_{N+1}$  obey the relation*

$$(\tilde{v}_{N+1} - v_{N+1})(\tilde{v}_N - v_N) = a^2 v_N \tilde{v}_N, \quad (\hat{v}_{N+1} - v_{N+1})(\hat{v}_N - v_N) = b^2 v_N \hat{v}_N, \quad (15)$$

*which is an auto BT of Q1(0).*

*Proof.* The first part of the theorem holds due to Lemma 3. For the second part, since  $(v_N, V_N)$  is a solution pair of the BT (12), they have compatibility  $\hat{\tilde{A}} = \tilde{\hat{A}}$ , and so does  $V_{N+1}$ . Then, on the basis of Lemma 2,  $v_{N+1}$  defined by (14b) solves Q1(0). For the relation (15), substituting (14a) into (14b) and using (12) with  $(v, V) = (v_N, V_N)$ , we arrive at (15). It provides an auto BT for Q1(0). In fact, there is a non-auto BT [20]

$$(u - \tilde{u})(v - \tilde{v}) = p(v\tilde{v} - \delta^2), \quad (u - \hat{u})(v - \hat{v}) = q(v\hat{v} - \delta^2), \quad (16)$$

to map  $v$  to  $u$  from Q1(0) to Q1( $\delta$ ). It holds as well for the reduced case  $\delta = 0$ , in which both  $v$  and  $u$  are solutions to Q1(0).  $\square$

### 3.2 Rational solutions of Q1(0) and lpmKdV

Theorem 1 describes a mechanism to generate new solutions for Q1(0) and the lpmKdV equation in an iterative way. If we start from a simple solution pair, e.g.  $(v_1 = an + bm + \gamma_1, V_1 = 1)$ , we can generate a sequence of rational solutions to Q1(0) and the lpmKdV equation. Some low order solutions in this sequence are

$$v_1 = x_1, \quad V_1 = 1, \quad (17a)$$

$$v_2 = \frac{1}{3}(x_1^3 - x_3), \quad V_2 = x_1, \quad (17b)$$

$$v_3 = \frac{1}{x_1} \left( \frac{1}{45}x_1^6 - \frac{1}{9}x_1^3x_3 + \frac{1}{5}x_1x_5 - \frac{1}{9}x_3^2 \right), \quad V_3 = \frac{x_1^3 - x_3}{3x_1}, \quad (17c)$$

$$v_4 = \frac{3}{x_1^3 - x_3} \left[ \frac{1}{4725}x_1^{10} - \frac{1}{315}x_1^7x_3 + \frac{1}{75}x_1^5x_5 - \frac{1}{27}x_1x_3^3 \right]$$

$$\begin{aligned}
& -\frac{1}{25}x_5^2 + \frac{1}{15}x_1^2x_3x_5 - \frac{1}{21}x_1^3x_7 + \frac{1}{21}x_3x_7, \\
V_4 = \frac{3}{x_1^3 - x_3} & \left[ \frac{1}{45}x_1^6 - \frac{1}{9}x_1^3x_3 + \frac{1}{5}x_1x_5 - \frac{1}{9}x_3^2 \right], \tag{17d}
\end{aligned}$$

where

$$x_i = a^i n + b^i m + \gamma_i, \quad \gamma_i \in \mathbb{C}, \quad (i = 1, 2, \dots). \tag{18}$$

We note that  $\{V_N\}$  are different from the rational solutions of the lpmKdV equation obtained in [18] as a reduction of the Hirata-Miwa equation.

In the following we prove that if we start from (17), all the solutions generated from (14) are meaningful. First, let us look at non-zero property.

**Lemma 4.** *Suppose  $a > 0, b > 0, v_N(0, 0) > 0$ , and we restrict  $(n, m)$  in the first quadrant  $\{n \geq 0, m \geq 0\}$ . Then  $v_N$  and  $V_N$  generated from (14) with (17a) satisfy  $v_N > 0, V_N > 0$ .*

*Proof.* Obviously, under assumption of the lemma, from (17a) we have  $v_1 > 0, V_1 > 0$  and  $V_2 = v_1/V_1 > 0$ . Then, suppose that  $v_N > 0, V_N > 0$  and consequently  $V_{N+1} = v_N/V_N > 0$ . Next, from (14b) we have

$$v_{N+1}(n+1, m) - v_{N+1}(n, m) > 0, \quad v_{N+1}(n, m+1) - v_{N+1}(n+1, m) > 0.$$

This implies

$$v_{N+1}(n, m) > v_{N+1}(n-1, m) > v_{N+1}(n-1, m-1) > \dots > v_{N+1}(0, 0).$$

If we take “integration” constant  $v_{N+1}(0, 0) > 0$ , then  $v_{N+1}(n, m)$  must be positive in quadrant  $\{n \geq 0, m \geq 0\}$ . □

Next, we observe that in  $v_1, v_2$  and  $v_3$  the order of leading terms (in terms of  $x_1$ ) are respectively 1, 3 and 5. Now we prove all the  $v_N$  defined through (14) with (17a) are distinct in the sense of having different leading orders in terms of  $x_1$ .

**Lemma 5.**  *$v_N$  has a leading order  $2N - 1$  in terms of  $x_1$  and  $V_N$  has a leading order  $N - 1$  in the same sense.*

*Proof.* From (17) we can suppose the lemma is correct up to some  $N$ . Then one can find

$$V_{N+1} = \frac{v_N}{V_N} \sim O(x_1^N),$$

and from (14b)

$$\tilde{v}_{N+1} - v_{N+1} = aV_N\tilde{V}_N \sim O(x_1^{2N}), \quad \hat{v}_{N+1} - v_{N+1} = bV_N\hat{V}_N \sim O(x_1^{2N}),$$

which means  $v_{N+1} \sim O(x_1^{2N+1})$ . Based on mathematical induction, the lemma holds. □

Thus, we conclude the following.

**Theorem 2.** *The iteration relation (14) is meaningful in generating distinct rational solutions from initial solutions (17a) for  $Q1(0)$  and the lpmKdV equation. These solutions are non-singular at least on the first quadrant  $\{n \geq 0, m \geq 0\}$  if we take  $a > 0, b > 0, v_N(0, 0) > 0$ .*

Note that not all solutions can be effectively iterated through (14). For example,

$$v_1 = \alpha^n \beta^m, \quad V_1 = v_1^{\frac{1}{2}}$$

are confined in (14) due to  $V_2 = v_1/V_1 = V_1$ . Here the parametrizations for  $a$  and  $b$  are  $a^2 = p = (1 - \alpha)^2/\alpha, b^2 = q = (1 - \beta)^2/\beta$ .

### 3.3 Solutions to Q1( $\delta$ )

The iteration (14) can be extended to  $N \leq 0$ .

**Lemma 6.** *Define*

$$v_{-N} = -\frac{1}{v_{N+1}}, \quad V_{-N} = (-1)^{N+1} \frac{1}{V_{N+2}}, \quad \text{for } N \geq 0. \quad (19)$$

*Then the iteration relation (14) is valid for all  $N \in \mathbb{Z}$ .*

This lemma can be checked directly.

Note that the extension does not lead to new solutions to Q1(0) and the lpmKdV equation because these two equations are invariant under transformations of type  $u \rightarrow c/u$ . However, the extension does bring more rational solutions to Q1( $\delta$ ).

**Theorem 3.** *For the pair  $(v_N, V_N)$  determined by (14) with  $N \in \mathbb{Z}$ , function*

$$u_N = v_N + \frac{\delta^2}{v_{N-2}}, \quad (N \in \mathbb{Z}) \quad (20)$$

*gives a sequence of solutions to Q1( $\delta$ ).  $u_N$  and  $v_{N-1}$  are connected via*

$$\tilde{u}_N - u_N = \frac{a^2(v_{N-1}\tilde{v}_{N-1} - \delta^2)}{\tilde{v}_{N-1} - v_{N-1}}, \quad \hat{u}_N - u_N = \frac{b^2(v_{N-1}\hat{v}_{N-1} - \delta^2)}{\hat{v}_{N-1} - v_{N-1}}, \quad (21)$$

*which is the non-auto BT (16) between Q1( $\delta$ ) and Q1(0). Besides, (20) agrees with the chain*

$$u_1 \xrightarrow{\delta=0} v_1 \xrightarrow{\text{BT (21)}} u_2 \xrightarrow{\delta=0} v_2 \xrightarrow{\text{BT (21)}} u_3 \cdots \cdots \cdots. \quad (22)$$

*Proof.* We only need to prove  $u_N$  defined by (20) satisfies (21). Since  $\{v_N\}$  obey the BT (15), using which we can find

$$\tilde{v}_k - v_k = \frac{a^2\tilde{v}_{k-1}v_{k-1}}{\tilde{v}_{k-1} - v_{k-1}}, \quad \frac{1}{\tilde{v}_{k-2}} - \frac{1}{v_{k-2}} = \frac{-a^2}{\tilde{v}_{k-1} - v_{k-1}}.$$

Then, from (20) by direct calculation we immediately reach

$$\tilde{u}_N - u_N = \tilde{v}_N - v_N + \delta^2 \left( \frac{1}{\tilde{v}_{N-2}} - \frac{1}{v_{N-2}} \right) = \frac{a^2(v_{N-1}\tilde{v}_{N-1} - \delta^2)}{\tilde{v}_{N-1} - v_{N-1}}.$$

Similarly we can find  $\hat{u}_N - u_N$  satisfies (21) as well. □

Formula (20) provides an explicit relation between solutions of  $Q1(\delta)$  and  $Q1(0)$ , where  $\{v_N\}$  is a sequence generated from (14). For  $v_N$  given as in (17), some rational solutions of  $Q1(\delta)$  generated from (20) are

$$u_1 = x_1 - \frac{1}{3}\delta^2(x_1^3 - x_3), \quad (23a)$$

$$u_2 = \frac{1}{3}(x_1^3 - x_3) - \delta^2 x_1, \quad (23b)$$

$$u_3 = \frac{1}{x_1} \left( \frac{1}{45}x_1^6 - \frac{1}{9}x_1^3 x_3 + \frac{1}{5}x_1 x_5 - \frac{1}{9}x_3^2 + \delta^2 \right), \quad (23c)$$

where we have made use of relation (19) in order to get  $v_{-1}$  and  $v_0$ .

Here we give two remarks.

*Remark 1.* There are some overlaps (dual forms) in the chain (20). Note that  $Q1(\delta)$  equation is formally invariant under transformation first replacing  $u$  with  $\varepsilon\delta^2 u$  and then  $\delta$  with  $1/\delta$  where  $\varepsilon = \pm 1$ , by which (20) is transformed into its dual form

$$u_N = \varepsilon \left( \delta^2 v_N + \frac{1}{v_{N-2}} \right), \quad (24)$$

which gives a sequence of solutions to  $Q1(\delta)$  as well. By the relation (19) given in Lemma 6,  $u_N$  and  $u_{3-N}$  in (20) are dual forms of each other.

*Remark 2.* Not all the rational solutions of  $Q1(\delta)$  are included in the chain (20). We give two exceptions. One is

$$u = \alpha n + \beta m + \gamma, \quad (25a)$$

where  $\gamma$  is a constant and  $\alpha, \beta$  are defined by parametrization

$$p = \frac{c_0}{a^2 - \delta^2}, \quad q = \frac{c_0}{b^2 - \delta^2}, \quad \alpha = pa, \quad \beta = qb, \quad (25b)$$

with arbitrary constant  $c_0$ , and the other is

$$u = \delta x_1^2 + \delta \gamma_0, \quad (26)$$

where  $x_1$  is defined as (18),  $p, q$  are parameterized as in (13) and  $\gamma_0$  is a constant.

### 3.4 Rational solutions to Q2

We make use of a non-auto BT between  $Q1(\delta)$  (6) and  $Q2$  (8) to derive rational solutions of  $Q2$ . The BT reads [20]

$$\delta(u - \tilde{u})(w - \tilde{w}) = p(2u\tilde{u} - \delta^2 w - \delta^2 \tilde{w}) + \delta p^2(u + \tilde{u} + \delta p), \quad (27a)$$

$$\delta(u - \hat{u})(w - \hat{w}) = q(2u\hat{u} - \delta^2 w - \delta^2 \hat{w}) + \delta q^2(u + \hat{u} + \delta q). \quad (27b)$$

When  $u$  is given by (25a) with parametrization (25b), from (27) we can find

$$w = \frac{u^2}{\delta^2} - \frac{c_0 u}{\delta^3} + \frac{c_0^2}{2\delta^4} + \left( \frac{a - \delta}{a + \delta} \right)^n \left( \frac{b - \delta}{b + \delta} \right)^m \gamma_0, \quad u = \alpha n + \beta m + \gamma. \quad (28)$$

where  $\gamma_0$  is a constant. This is not a pure rational solution.

For  $u$  defined in (26) with parametrization (13), from (27) we find

$$w = \frac{1}{5}x_1^4 + \frac{2}{3}\gamma_0x_1^2 + \frac{4\gamma_0x_3}{3x_1} + \frac{4x_5}{5x_1} + \gamma_0^2. \quad (29)$$

For  $u = u_2$  given by (23b) with parametrization (13), from (27) we find

$$w = \frac{1}{45\delta(x_1 - \delta)} \left[ 30\delta^3(x_1^3 - x_3) - 15\delta^2(x_1^4 + 2x_1x_3) - 3\delta(x_1^5 - 10x_1^2x_3 - 6x_5) \right. \\ \left. + 2x_1^6 + 18x_1x_5 - 10x_3^2 - 10x_1^3x_3 \right]. \quad (30)$$

### 3.5 Solutions to H3( $\delta$ )

To obtain solutions to H3( $\delta$ ) we make use of A1( $\delta$ ) (7). Solutions to A1( $\delta$ ) can be obtained from those of Q1( $\delta$ ) (6) through transformation [5]

$$z = (-1)^{n+m}u. \quad (31)$$

Similar to the triplet composed by (11), (3) and (12), A1( $\delta$ ) (7) with parametrization (13), H3( $\delta$ ) (4) and their non-auto BT [20]

$$\tilde{z} + z - \delta a^2 = aZ\tilde{Z}, \quad (32a)$$

$$\hat{z} + z - \delta b^2 = bZ\hat{Z} \quad (32b)$$

constitute a consistent triplet, i.e. compatibility  $\hat{\tilde{z}} = \tilde{\hat{z}}$  in (32) requires  $Z$  satisfies (4) and  $\tilde{\tilde{Z}} = \hat{\hat{Z}}$  requires  $z$  satisfies (7). Such a consistency leads to

**Lemma 7.** (1). Starting from any solution  $z$  of (7), by integration through (32), the resulted  $Z$  solves (4), and vice versa; (2). any solution pair  $(z, Z)$  of (32) gives a solution  $z$  to (7) and  $Z$  to (4).

Solution sequences of A1( $\delta$ ) and H3( $\delta$ ) are then given as follows.

**Theorem 4.** For  $v_N$  and  $V_N$  constructed in Theorem 1 and Lemma 6, function

$$z_N = (-1)^{n+m} \left( v_N + \frac{\delta^2}{v_{N-2}} \right), \quad (N \in \mathbb{Z}) \quad (33)$$

solves A1( $\delta$ ) (7), and

$$Z_N = (-1)^{\frac{n+m}{2} + \frac{1}{4}} \left( V_N + \frac{(-1)^{n+m}\delta}{V_{N-1}} \right), \quad (N \in \mathbb{Z}) \quad (34)$$

solves H3( $\delta$ ) (4).

*Proof.* Since  $u_N$  defined in (20) solves Q1( $\delta$ ), it is obvious that (33) provides a solution to A1( $\delta$ ). Besides, making use of iterative relation (14), one can find (33) and (34) provide a solution pair to (32), which proves the present theorem.  $\square$



Here we list some solutions for H3( $\delta$ ),

$$\begin{aligned} Z_1 &= (-1)^{\frac{n+m}{2}+\frac{1}{4}} (1 - (-1)^{n+m} \delta x_1), \\ Z_2 &= (-1)^{\frac{n+m}{2}+\frac{1}{4}} (x_1 + (-1)^{n+m} \delta), \\ Z_3 &= (-1)^{\frac{n+m}{2}+\frac{1}{4}} \frac{x_1^3 - x_3 + 3(-1)^{n+m} \delta}{3x_1}. \end{aligned}$$

Similar to Q1( $\delta$ ), there are also overlaps (dual forms) in the chain (34) for H3( $\delta$ ). H3( $\delta$ ) is formally invariant under transformation first  $Z \rightarrow \varepsilon \delta^{-1} (-1)^{n+m} Z$  and then  $\delta \rightarrow -\delta^{-1}$  with  $\varepsilon = \pm 1$ , by which (34) becomes

$$Z_N = \varepsilon (-1)^{\frac{n+m}{2}+\frac{1}{4}} \left( \delta (-1)^{n+m} V_N - \frac{1}{V_{N-1}} \right).$$

$Z_N$  and  $Z_{3-N}$  in (34) are dual forms of each other in light of relation (19).

## 4 Solutions of H1 and H2

In this section we first derive solutions of H1 using a relation between H1 and the lpmKdV equation. Then from H1 we derive solutions of H2.

### 4.1 H1

There is a non-auto BT [1]

$$\tilde{u} - \hat{u} = \frac{b\tilde{V} - a\hat{V}}{abV}, \quad (35a)$$

$$\hat{\tilde{u}} - u = \frac{b\hat{\tilde{V}} + aV}{ab\hat{V}} \quad (35b)$$

to connect H1(u) (1) and the lpmKdV(V) equation (3). We use it to derive solutions for H1 on the basis of the following fact.

**Lemma 8.** *When  $V$  solves the lpmKdV equation (3), function  $u$  defined by (35) satisfies H1 (1) with parametrization*

$$p = -1/a^2, \quad q = -1/b^2. \quad (36)$$

*Proof.* We rewrite the lpmKdV equation (3) as

$$(b\tilde{V} - a\hat{V})(b\hat{\tilde{V}} + aV) = (b^2 - a^2)V\hat{V}. \quad (37)$$

Then, multiplying both equations in (35) and making use of (37) we immediately reach H1 (1) provided  $p, q$  are parameterized as (36). □

From (35) we find

$$\tilde{\tilde{u}} - u = \frac{V + \tilde{\tilde{V}}}{a\tilde{\tilde{V}}}, \quad \hat{\hat{u}} - u = \frac{V + \hat{\hat{V}}}{b\hat{\hat{V}}}, \quad (38)$$

of which we make use to derive  $u$  from known  $V$ .

For  $V_1 = 1$  and  $V_2 = x_1$  given in (17), we derive same solution for H1,

$$u_1 = u_2 = x_{-1}, \quad (39a)$$

where  $x_{-1}$  follows the definition (18) with  $i = -1$ . For  $V_3$  and  $V_4$  given in (17), we respectively find

$$u_3 = x_{-1} - \frac{1}{x_1}, \quad (39b)$$

and

$$u_4 = x_{-1} - \frac{3x_1^2}{x_1^3 - x_3}. \quad (39c)$$

Note that the lpmKdV equation (3) is invariant under  $V \rightarrow \frac{1}{V}$ . So we can replace  $V$  by  $1/V$  in (38) and get

$$\tilde{\tilde{u}} - u = \frac{\tilde{V}}{a} \left( \frac{1}{V} + \frac{1}{\tilde{\tilde{V}}} \right), \quad \hat{\hat{u}} - u = \frac{\hat{V}}{b} \left( \frac{1}{V} + \frac{1}{\hat{\hat{V}}} \right).$$

One may wonder if the above relation can be used to obtain more solutions for H1. However, making use of iterative relations (14) we find

$$\tilde{\tilde{u}}_{N+1} - u_{N+1} = \frac{V_{N+1} + \tilde{\tilde{V}}_{N+1}}{a\tilde{\tilde{V}}_{N+1}} = \frac{\tilde{V}_N}{a} \left( \frac{1}{V_N} + \frac{1}{\tilde{\tilde{V}}_N} \right),$$

and a same formula for  $(\hat{\hat{u}}, b)$ , which means  $V \rightarrow \frac{1}{V}$  does not lead new solutions for H1. This can also explain the fact  $u_1 = u_2$  due to  $V_1 = \frac{1}{V_1} = 1$ .

## 4.2 H2

Again, H2 (2), H1 (1) and their non-auto BT [20]

$$v + \tilde{v} + p = 2u\tilde{u}, \quad (40a)$$

$$v + \hat{v} + q = 2u\hat{u} \quad (40b)$$

constitute a consistent triplet with parametrization (36). Then, from solutions (39) of H1 and BT (40), we find the following rational solutions for H2:

$$v_1 = v_2 = x_{-1}^2, \quad (41a)$$

$$v_3 = x_{-1}^2 - \frac{2x_{-1}}{x_1}, \quad (41b)$$

$$v_4 = x_{-1}^2 - \frac{6x_1(x_1x_{-1} - 1)}{x_1^3 - x_3}. \quad (41c)$$

## 5 Rational solutions in determinant form

From the previous section it is understood that the sequence  $\{V_N\}$  plays a crucial role in constructing solutions in the whole paper. With regard to rational solutions, it is hard to do “integration” from (14b) to get high order  $v_N$  and consequently it is difficult to get high order  $V_N$ . In this section we aim to construct Casoratian expressions for  $v_N$  and  $V_N$ , as well as rational solutions of other equations.

### 5.1 Bilinear relation of $V_N$ and $v_N$

We express

$$V_N = \frac{P_{N-1}}{P_{N-2}} \quad (42a)$$

and it then follows from (14a) that

$$v_N = \frac{P_N}{P_{N-2}}. \quad (42b)$$

From  $V_2 = x_1$  we introduce

$$P_0 = 1, \quad P_1 = x_1, \quad (43a)$$

and from (17) we find successively

$$P_2 = \frac{x_1^3 - x_3}{3}, \quad (43b)$$

$$P_3 = \frac{1}{45}x_1^6 - \frac{1}{9}x_1^3x_3 + \frac{1}{5}x_1x_5 - \frac{1}{9}x_3^2, \quad (43c)$$

$$P_4 = \frac{1}{4725}x_1^{10} - \frac{1}{315}x_1^7x_3 + \frac{1}{75}x_1^5x_5 - \frac{1}{27}x_1x_3^3 - \frac{1}{25}x_5^2 + \frac{1}{15}x_1^2x_3x_5 - \frac{1}{21}x_1^3x_7 + \frac{1}{21}x_3x_7, \quad (43d)$$

where  $P_4$  is obtained from the relation  $\frac{P_4}{P_3} = V_5 = \frac{v_4}{V_4}$ .

Viewing (42) as transformations, the BT (14b) yields

$$\widetilde{\overline{P}} \underline{P} - \overline{P} \widetilde{\underline{P}} = aP\widetilde{P}, \quad (44a)$$

$$\widehat{\overline{P}} \underline{P} - \overline{P} \widehat{\underline{P}} = bP\widehat{P}, \quad (44b)$$

where

$$P \doteq P_N, \quad \overline{P} \doteq P_{N+1}, \quad \underline{P} \doteq P_{N-1}.$$

This is a bilinear equation set for polynomials  $\{P_N\}$ . Note that based on (19), the relations (42) and (44) can be generated to  $N \in \mathbb{Z}$  by defining

$$P_{-N} = (-1)^{[\frac{N}{2}]} P_{N-1}, \quad (45)$$

where  $[\cdot]$  denotes the greatest integer function.

In Sec.5.3 we will give a Casoratian form of  $P$ . To achieve that, we make use of H1.

### 5.2 Casoratian form of rational solutions of H1

For H1 (1), using 3D consistency we have its BT

$$(\widetilde{u} - \overline{u})(\widetilde{\overline{u}} - u) = a^{-2} - k^{-2}, \quad (46a)$$

$$(\widehat{u} - \overline{u})(\widehat{\overline{u}} - u) = b^{-2} - k^{-2}, \quad (46b)$$

where  $\overline{u}$  stands for a new solution of H1, we adopt parametrization (36) and the arbitrary number  $k^{-2} = r$  acts as a “soliton number” which leads to a new soliton (cf. [11]). Now we remove the term  $k^{-2}$  from (46), i.e. taking  $r = 0$ , and consequently we have

$$(\widetilde{u} - \overline{u})(\widetilde{\overline{u}} - u) = a^{-2}, \quad (47a)$$

$$(\widehat{u} - \overline{u})(\widehat{u} - u) = b^{-2}, \quad (47b)$$

which can generate a rational part in the new solution  $\overline{u}$ .

To find solutions from (47), first, we introduce

$$\widetilde{u} - \overline{u} = \frac{f \widetilde{f}}{a \widetilde{f} \overline{f}}, \quad (48a)$$

$$\widetilde{\overline{u}} - u = \frac{\widetilde{f} \overline{f}}{a \widetilde{f} \overline{f}}, \quad (48b)$$

$$\widehat{u} - \overline{u} = \frac{f \widehat{f}}{b \widehat{f} \overline{f}}, \quad (48c)$$

$$\widehat{\overline{u}} - u = \frac{\widehat{f} \overline{f}}{b \widehat{f} \overline{f}}, \quad (48d)$$

which provide a factorization of (47). Such an assumption coincides with the previous results. In fact, suppose  $V = \overline{f}/f$ , then from (48) we can find  $\widetilde{u} - \widehat{u}$  and  $\widehat{\overline{u}} - u$  agree with (35) and  $\widetilde{u} - u$  and  $\widehat{\overline{u}} - u$  agree with (38). Then we introduce

$$u = x_{-1} - \frac{g}{f}, \quad \overline{u} = x_{-1} - \frac{\overline{g}}{\overline{f}}, \quad (49)$$

by which we bilinearize (48) as

$$\overline{g} \widetilde{f} - \overline{f} \widetilde{g} + \frac{1}{a}(\overline{f} \widetilde{f} - \widetilde{f} f) = 0, \quad (50a)$$

$$g \widetilde{\overline{f}} - f \widetilde{\overline{g}} - \frac{1}{a}(\overline{f} \widetilde{f} - \widetilde{f} f) = 0, \quad (50b)$$

$$\overline{g} \widehat{f} - \overline{f} \widehat{g} + \frac{1}{b}(\overline{f} \widehat{f} - \widehat{f} f) = 0, \quad (50c)$$

$$g \widehat{\overline{f}} - f \widehat{\overline{g}} - \frac{1}{b}(\overline{f} \widehat{f} - \widehat{f} f) = 0. \quad (50d)$$

Next, we introduce Casoratian forms for  $f, \overline{f}, g$  and  $\overline{g}$ . Consider function

$$\psi_i(n, m, l) = \psi_i^+(n, m, l) + \psi_i^-(n, m, l), \quad \psi_i^\pm(n, m, l) = \varrho_i^\pm (1 \pm s_i)^{l+l_0} (1 \pm a s_i)^n (1 \pm b s_i)^m, \quad (51)$$

where  $\varrho_i^\pm$  and  $s_i$  are nonzero constants<sup>1</sup>. This can be used to construct soliton solutions for H1 equation (cf. [11])<sup>2</sup>. To obtain rational solutions, we expand  $\psi_i^\pm(n, m, l)$  as

$$\psi_i^\pm(n, m, l) = \pm \frac{1}{2} \sum_{h=0}^{\infty} \alpha_h^\pm s_i^h, \quad (52a)$$

where

$$\alpha_h^\pm = (\pm 1)^h \sum_{j=0}^h \sum_{k=0}^j C_{l+l_0}^k C_m^{j-k} C_n^{h-j} a^{h-j} b^{j-k}, \quad (52b)$$

<sup>1</sup>In the paper we suppose  $l_0$  is either a large enough integer or a non-integer so that the derivative  $\partial_{s_i}^h (1 \pm s_i)^{l+l_0}|_{s_i=0} \neq 0$ .

<sup>2</sup>One needs to use gauge property of bilinear H1 and make certain extension from  $(\pm s_i)^{l+l_0}$  to  $(1 \pm s_i)^{l+l_0}$ .

we have taken  $\varrho_i^\pm = \pm \frac{1}{2}$  and  $C_n^s$  is defined as

$$C_n^s = \begin{cases} \frac{n!}{s!(n-s)!}, & 0 \leq s \leq n, \\ 1, & s = 0, \\ 0, & 0 \leq n < s, \\ \frac{n(n-1)\cdots(n-s+1)}{s!}, & n < 0, s > 0. \end{cases}$$

Here we note that noticing that

$$\psi_i^\pm(n, m, l) = \varrho_i^\pm \exp \left[ \pm \sum_{j=1}^{\infty} \frac{1}{j} (\mp 1)^{j+1} x_j s^j \right]$$

where  $x_j$  are defined as (18) but all  $\gamma_j = l + l_0$ , all  $\{\alpha_h^\pm\}$  can be expressed in terms of  $\{x_j\}$ . For  $\{\alpha_h^+\}$  we have

$$\alpha_h^+ = (-1)^h \sum_{||\mu||=h} (-1)^{|\mu|} \frac{\mathbf{x}^\mu}{\mu!} \quad (53)$$

where

$$\mu = (\mu_1, \mu_2, \dots), \quad ||\mu|| = \sum_{j=1}^{\infty} j \mu_j, \quad |\mu| = \sum_{j=1}^{\infty} \mu_j, \quad \mu! = \mu_1! \cdot \mu_2! \cdots, \quad \mathbf{x}^\mu = \left(\frac{x_1}{1}\right)^{\mu_1} \left(\frac{x_2}{2}\right)^{\mu_2} \cdots,$$

and  $\mu_j \in \{0, 1, 2, \dots\}$ . The first few  $\alpha_h^+$  are

$$\begin{aligned} \alpha_0^+ &= 1, \quad \alpha_1^+ = x_1, \quad \alpha_2^+ = \frac{1}{2}(x_1^2 - 2x_2), \quad \alpha_3^+ = \frac{1}{6}(x_1^3 - 3x_1x_2 + 2x_3), \\ \alpha_4^+ &= \frac{1}{24}(x_1^4 - 6x_1^2x_2 + 8x_1x_3 + 3x_2^2 - 6x_4), \\ \alpha_5^+ &= \frac{1}{120}(x_1^5 - 10x_2x_1^3 + 20x_3x_1^2 + 15x_2^2x_1 - 30x_4x_1 - 20x_2x_3 + 24x_5). \end{aligned}$$

Introduce a column vector

$$\alpha(n, m, l) = (\alpha_0, \alpha_1, \dots, \alpha_{N-1})^T, \quad \alpha_j = \alpha_{2j+1}^+. \quad (54)$$

With  $\alpha(n, m, l)$  as a basic column vector we introduce Casoratians w.r.t. shifts in  $l$ :

$$f = |\widehat{N-1}|_R = |\alpha(n, m, 0), \alpha(n, m, 1), \dots, \alpha(n, m, N-1)|, \quad (55a)$$

$$\bar{f} = |\widehat{N}|_R, \quad g = |\widehat{N-2}, N|_R - Nf, \quad \bar{g} = |\widehat{N-1}, N+1|_R - (N+1)\bar{f}. \quad (55b)$$

Some  $f, g$  of low orders are

$$f_{N=1} = x_1, \quad g_{N=1} = 1, \quad (56a)$$

$$f_{N=2} = \frac{x_1^3 - x_3}{3}, \quad g_{N=2} = x_1^2, \quad (56b)$$

$$f_{N=3} = \frac{1}{45}x_1^6 - \frac{1}{9}x_1^3x_3 + \frac{1}{5}x_1x_5 - \frac{1}{9}x_3^2, \quad g_{N=3} = \frac{2}{15}x_1^5 - \frac{1}{3}x_1^2x_3 + \frac{1}{5}x_5, \quad (56c)$$

where all  $\gamma_i = l_0$  in  $x_i$ . Through (49),  $(f, g)$  with  $N = 1, 2$  provide solutions (39b) and (39c) for H1. For general  $N$ , we have the following.

**Theorem 5.** *The Casoratians (55) solve the bilinear BT (50) and (49) provides rational solutions to H1.*

Proof will be given in Appendix A.

*Remark* There is an alternative choice for the Casoratians (55), which are given by just replacing the basic column vector  $\alpha$  given in (54) with

$$\beta(n, m, l) = (\beta_0, \beta_1, \dots, \beta_{N-1})^T, \quad \beta_j = \alpha_{2j}^+, \quad (57)$$

where  $\alpha_{2j}^+$  are defined in (52b), or equivalently,

$$\beta_j = \frac{1}{(2j)!} \partial_{s_i}^{2j} \psi_i|_{s_i=0}, \quad (\varrho_i^\pm = \frac{1}{2}). \quad (58)$$

### 5.3 Casoratian solutions to (44)

We can make use of the BT of H1 to obtain solutions to bilinear equation (44). By the compatibility of (48a) and (48b), i.e.  $(E_n - E_N)(E_n E_N - 1)u = (E_n E_N - 1)(E_n - E_N)u$  where  $E_N f = \bar{f}$ , we find

$$\frac{\tilde{\tilde{f}}f - \bar{f}\tilde{f}}{\tilde{\tilde{f}}\bar{f}} = E_n \left( \frac{\tilde{\tilde{f}}f - \bar{f}\tilde{f}}{\tilde{\tilde{f}}\bar{f}} \right).$$

Similarly,

$$\frac{\hat{\hat{f}}f - \bar{f}\hat{f}}{\hat{\hat{f}}\bar{f}} = E_m \left( \frac{\hat{\hat{f}}f - \bar{f}\hat{f}}{\hat{\hat{f}}\bar{f}} \right).$$

This means

$$\tilde{\tilde{f}}f - \bar{f}\tilde{f} = \lambda_1(m, N)\tilde{\tilde{f}}\bar{f}, \quad \hat{\hat{f}}f - \bar{f}\hat{f} = \lambda_2(n, N)\hat{\hat{f}}\bar{f}. \quad (59)$$

Next we go to prove  $\lambda_1(m, N) = a$  and  $\lambda_2(n, N) = b$ . Again, from (48), we can derive

$$\bar{u} - u = \frac{\tilde{\tilde{f}}f}{a\tilde{\tilde{f}}\bar{f}} - \frac{\bar{f}\tilde{f}}{a\tilde{\tilde{f}}\bar{f}} = \frac{\tilde{\tilde{f}}f}{b\hat{\hat{f}}\bar{f}} - \frac{\bar{f}\hat{f}}{b\hat{\hat{f}}\bar{f}}.$$

Using (59) to eliminate  $\tilde{\tilde{f}}$  and  $\hat{\hat{f}}$  from the above equation, we find

$$\bar{u} - u = -\lambda_1(m, N)\frac{\bar{f}^2}{a\tilde{\tilde{f}}\bar{f}} = -\lambda_2(n, N)\frac{\bar{f}^2}{b\hat{\hat{f}}\bar{f}}, \quad (60)$$

which means

$$a\lambda_2(n, N) = b\lambda_1(m, N),$$

and it then follows that both  $\lambda_1$  and  $\lambda_2$  must be  $(n, m)$ -independent. We assume

$$\gamma(N) = \lambda_1/a = \lambda_2/b,$$

and then (59) yields

$$\gamma(N) = \frac{\tilde{\tilde{f}}f - \bar{f}\tilde{f}}{a\tilde{\tilde{f}}\bar{f}} = \frac{\hat{\hat{f}}f - \bar{f}\hat{f}}{b\hat{\hat{f}}\bar{f}}. \quad (61)$$

To determine the value of  $\gamma(N)$ , we investigate properties of  $f$  near the point  $(n, m) = (0, 0)$ , which are presented through the following lemmas.

**Lemma 9.** According to the definitions of  $u$  in (49),  $f$  and  $g$  in (55) and  $\alpha_h^\pm$  in (52b), we find the value of  $\alpha_h^\pm|_{(n,m)=(0,0)}$  is independent of  $(a,b)$ , and so are  $f(0,0)$ ,  $g(0,0)$  and  $u(0,0)$ . Then, from (60) we find that  $\gamma(N)$  must be independent of  $(a,b)$ .

**Lemma 10.** For same  $N$ , at point  $(n,m) = (0,0)$  there exists relation

$$f_N(\alpha(n,m,l)) = f_{N+1}(\beta(n,m,l)), \quad (62)$$

where  $\alpha(n,m,l)$  and  $\beta(n,m,l)$  are respectively  $N$ -th order and  $(N+1)$ -th order column vectors defined as (54) and (57). Here and below  $f_N(\psi)$  stands for a  $N$ -th order Casoratian  $|\widehat{N-1}|$  composed by a  $N$ -th order basic column vector  $\psi$ .

*Proof.* For convenience we introduce vector notations

$$\xi(l; N) = \alpha(0,0,l) = (\xi_0(l), \xi_1(l), \dots, \xi_{N-1}(l))^T, \quad \xi_j(l) = C_{l+l_0}^{2j+1}, \quad (63a)$$

$$\zeta(l; N) = \beta(0,0,l) = (\zeta_0(l), \zeta_1(l), \dots, \zeta_{N-1}(l))^T, \quad \zeta_j(l) = C_{l+l_0}^{2j}. \quad (63b)$$

Using combinatorial formula  $C_x^y = C_{x-1}^y + C_{x-1}^{y-1}$  where  $x \in \mathbb{R}$  and  $y \in \mathbb{Z}^+$ , we have

$$\zeta(l+1; N+1) - \zeta(l; N+1) = \begin{pmatrix} 0 \\ \xi(l; N) \end{pmatrix},$$

which leads to the relation

$$f_{N+1}(\zeta(l; N+1)) = f_N(\xi(l; N)), \quad (64)$$

i.e. (62). □

**Lemma 11.** For Casoratian  $f_N(\alpha(n,m,l))$ , the relation

$$f_N(\alpha(1,0,l)) = a^N f_{N-1}(\alpha(0,0,l)) + O(a^{N-1})$$

holds.

*Proof.*

$$\begin{aligned} f_N(\alpha(1,0,l)) &= f_N(a\zeta(l; N) + \xi(l; N)) = a^N f_N(\zeta(l; N)) + O(a^{N-1}) \\ &= a^N f_{N-1}(\xi(l; N-1)) + O(a^{N-1}), \end{aligned}$$

where we have made use of relation (64). □

With this lemma, for  $f = f_N(\alpha(n,m,l))$  in (61), we have

$$\begin{aligned} \widetilde{f}|_{n=m=0} &= a^N \underline{f}|_{n=m=0} + O(a^{N-1}), \quad \widetilde{\widetilde{f}}|_{n=m=0} = a^{N+1} f|_{n=m=0} + O(a^N), \\ \widetilde{\widetilde{\widetilde{f}}}|_{n=m=0} &= a^{N+2} \overline{f}|_{n=m=0} + O(a^{N+1}). \end{aligned}$$

Then, since  $\gamma(N)$  is independent of  $a$ , from (61) we arrive at

$$\gamma(N) = \lim_{a \rightarrow \infty} \frac{\widetilde{\widetilde{f}}f - \overline{\widetilde{f}}\widetilde{f}}{a\widetilde{\widetilde{f}}\overline{f}} \Big|_{n=m=0} = 1.$$

We can sum up this subsection with the following theorem.

**Theorem 6.** *The Casoratian  $f = f_N(\alpha(n, m, l))$  solves bilinear equation set*

$$\widetilde{\widetilde{f}}f - \overline{\widetilde{f}}\widetilde{f} = a\widetilde{f}\overline{f}, \quad (65a)$$

$$\widehat{\widehat{f}}f - \overline{\widehat{f}}\widehat{f} = b\widehat{f}\overline{f}. \quad (65b)$$

$P = f_N(\alpha(n, m, l))$  provides a Casoratian form of solution to (44). Same as (45), by defining

$$f_{-N} = (-1)^{[\frac{N}{2}]} f_{N-1}, \quad f_0 = 1, \quad (66)$$

one can consistently extend (65) to  $N \in \mathbb{Z}$ .

#### 5.4 Casoratian rational solutions to H2 and a sum-up

We can derive Casoratian rational solutions for H2 through non-auto BT (40), in which we suppose

$$u = x_{-1} - \frac{g}{f}, \quad v = x_{-1}^2 - 2(x_{-1} + N)\frac{g}{f} + \frac{h}{f} - N^2. \quad (67)$$

Then BT (40) is bilinearized as

$$f\widetilde{h} + \widetilde{f}h + 2(a^{-1} - N)g\widetilde{f} - 2(a^{-1} + N)\widetilde{g}f - 2g\widetilde{g} - 2N^2f\widetilde{f} = 0, \quad (68a)$$

$$f\widehat{h} + \widehat{f}h + 2(b^{-1} - N)g\widehat{f} - 2(b^{-1} + N)\widehat{g}f - 2g\widehat{g} - 2N^2f\widehat{f} = 0. \quad (68b)$$

Based on the bilinear form we have

**Theorem 7.** *The Casoratians*

$$f = |\widehat{N-1}|_R, \quad g = |\widehat{N-2}, N|_R - Nf, \quad h = |\widehat{N-2}, N+1|_R + |\widehat{N-3}, N-1, N|_R \quad (69)$$

composed by  $\alpha$  (54) solve the bilinear BT (68). Then, (67) provides rational solutions to H1 and H2.

Proof will be given in Appendix B.

Besides (56), some  $h$  of low orders are

$$h_{N=1} = x_1 + 2, \quad (70a)$$

$$h_{N=2} = \frac{4}{3}(x_1^3 - x_3) + 4x_1^2 + 2x_1, \quad (70b)$$

$$h_{N=3} = \frac{1}{5}x_1^6 - x_1^3x_3 + \frac{9}{5}x_1x_5 - x_3^2 + \frac{4}{5}x_1^5 - 2x_1^2x_3 + \frac{6}{5}x_5 + \frac{2}{3}x_1^4 - \frac{2}{3}x_1x_3, \quad (70c)$$

where all  $\gamma_i = l_0$  in  $x_i$ .

So far we have obtained Casoratian expressions for the rational solutions of Q1(0), lpmKdV, Q1( $\delta$ ), H3( $\delta$ ), H1 and H2. Noting that all these solutions are related to the rational solutions  $V_N$  of the lpmKdV equation, it is necessary to express all these obtained solutions through the Casoratians with a unified  $N$ . We collect them in the following theorem.

**Theorem 8.** *Suppose that*

$$f \doteq f_N = |\widehat{N-1}|_R, \quad g \doteq g_N = |\widehat{N-2}, N|_R - Nf, \quad (71a)$$



$$h \doteq h_N = |\widehat{N-2}, N+1|_R + |\widehat{N-3}, N-1, N|_R, \quad (71b)$$

and denote  $\overline{f} = f_{N+1}$  and  $\underline{f} = f_{N-1}$ . Then the rational solutions for  $Q1(0)$ ,  $lpmKdV$ ,  $Q1(\delta)$ ,  $H3(\delta)$ ,  $H1$  and  $H2$  are respectively

$$Q1(0) : v_{N+2} = \frac{\overline{\overline{f}}}{f}, \quad (72a)$$

$$lpmKdV : V_{N+2} = \frac{\overline{f}}{f}, \quad (72b)$$

$$Q1(\delta) : u_{N+2} = \frac{\overline{\overline{f}} + \delta^2 \underline{\underline{f}}}{f}, \quad (72c)$$

$$H3(\delta) : Z_{N+2} = (-1)^{\frac{n+m}{2} + \frac{1}{4}} \frac{\overline{f} + (-1)^{n+m} \delta \underline{\underline{f}}}{f}, \quad (72d)$$

$$H1 : u_{N+2} = x_{-1} - \frac{g}{f}, \quad (72e)$$

$$H2 : v_{N+2} = x_{-1}^2 - 2(x_{-1} + N) \frac{g}{f} + \frac{h}{f} - N^2. \quad (72f)$$

## 5.5 Rational solutions to Q2

Now we come to the final equation, Q2. We start from the non-auto BT (27) in which we take parametrization (13) and  $u$  to be (72c) which is a solution of  $Q1(\delta)$ . Introduce auxiliary function

$$w = y + \frac{u^2}{\delta^2}$$

by which the BT (27) yields

$$\widetilde{y} = \frac{\widetilde{u} - u - \delta p}{\widetilde{u} - u + \delta p} y + \frac{1}{\delta^2} (u + \delta p - \widetilde{u})(u + \delta p + \widetilde{u}), \quad (73a)$$

$$\widehat{y} = \frac{\widehat{u} - u - \delta q}{\widehat{u} - u + \delta q} y + \frac{1}{\delta^2} (u + \delta q - \widehat{u})(u + \delta q + \widehat{u}). \quad (73b)$$

Then, making use of the relation (65), from (72c) we can find

$$\begin{aligned} \widetilde{u}_{N+2} - u_{N+2} + \delta p &= a \frac{(\overline{f} + \delta \underline{\underline{f}})(\widetilde{\overline{f}} - \delta \widetilde{\underline{\underline{f}}})}{f \widetilde{f}}, \\ \widetilde{u}_{N+2} - u_{N+2} - \delta p &= a \frac{(\overline{f} - \delta \underline{\underline{f}})(\widetilde{\overline{f}} + \delta \widetilde{\underline{\underline{f}}})}{f \widetilde{f}}. \end{aligned}$$

On the basis of the above relations together with their  $(q, \wedge)$  version, and introducing

$$\theta_{N+2} = y_{N+2} \frac{\overline{f} - \delta \underline{\underline{f}}}{\widetilde{\overline{f}} + \delta \widetilde{\underline{\underline{f}}}},$$

we then reduce (73) to

$$\theta_{N+2} - \widetilde{\theta}_{N+2} = \frac{a(\overline{f} - \delta \underline{\underline{f}})(\widetilde{\overline{f}} - \delta \widetilde{\underline{\underline{f}}})}{\delta^2 f \widetilde{f}} (u_{N+2} + \widetilde{u}_{N+2} + \delta p), \quad (74a)$$

$$\theta_{N+2} - \hat{\theta}_{N+2} = \frac{b(\bar{f} - \delta \underline{f})(\hat{\bar{f}} - \delta \hat{\underline{f}})}{\delta^2 \underline{f} \hat{\underline{f}}}(u_{N+2} + \hat{u}_{N+2} + \delta q). \quad (74b)$$

To solve this system we expand

$$\theta_{N+2} = \sum_{i=-2}^2 \theta_{N+2}^{(i)} \delta^i.$$

It then follows from (74) that

$$\theta_{N+2}^{(-2)} - \tilde{\theta}_{N+2}^{(-2)} = \frac{a \tilde{\underline{f}} \bar{\underline{f}}}{f^2 \tilde{f}^2} (\tilde{\underline{f}} \bar{\underline{f}} + \tilde{f} \bar{\underline{f}}), \quad (75a)$$

$$\theta_{N+2}^{(-1)} - \tilde{\theta}_{N+2}^{(-1)} = \frac{-2a}{f^2 \tilde{f}^2} (\tilde{f} \underline{\tilde{f}} \tilde{\underline{f}} f + \tilde{\underline{f}} \underline{\tilde{f}} \tilde{f} \bar{\underline{f}}), \quad (75b)$$

$$\theta_{N+2}^{(0)} - \tilde{\theta}_{N+2}^{(0)} = \frac{a^2}{f^2 \tilde{f}^2} (\tilde{\underline{f}}^2 \bar{\underline{f}}^2 - \underline{f}^2 \tilde{\underline{f}}^2) + \frac{2(\tilde{\underline{f}} \bar{\underline{f}} - \tilde{f} \bar{\underline{f}})}{f \tilde{f}}, \quad (75c)$$

$$\theta_{N+2}^{(1)} - \tilde{\theta}_{N+2}^{(1)} = \frac{-2a}{f^2 \tilde{f}^2} (f \underline{\tilde{f}} \tilde{\underline{f}} \tilde{f} + \tilde{\underline{f}} \underline{\tilde{f}} \tilde{f} \bar{\underline{f}}), \quad (75d)$$

$$\theta_{N+2}^{(2)} - \tilde{\theta}_{N+2}^{(2)} = \frac{a \tilde{\underline{f}} \underline{f}}{f^2 \tilde{f}^2} (\tilde{\underline{f}} \underline{f} + \tilde{f} \underline{\underline{f}}), \quad (75e)$$

among which, except  $\theta_{N+2}^{(0)}$ , we find explicit expressions for  $\theta_{N+2}^{(i)}$  in terms of  $f$ :

$$\theta_{N+2}^{(-1)} = -\frac{\bar{\underline{f}}^2}{f^2}, \quad \theta_{N+2}^{(-1)} = \frac{2\bar{\underline{f}}^2 \underline{f} + 2f^2 \bar{\underline{f}}}{f^2 \underline{f}}, \quad \theta_{N+2}^{(1)} = -\frac{2f^2 \bar{\underline{f}} + 2f^2 \underline{f}}{f^2 \underline{f}}, \quad \theta_{N+2}^{(2)} = \frac{f^2}{\underline{\underline{f}}^2}. \quad (76)$$

For  $\theta_{N+2}^{(0)}$  which is determined by (75c), the simplest two items are

$$\theta_2^{(0)} = \frac{1}{3}x_1^4 + \frac{2}{3}x_1x_3, \quad \theta_3^{(0)} = -\frac{1}{15}x_1^4 + \frac{2}{3}x_1x_3 + \frac{2x_5}{5x_1}.$$

However, so far we do not find an explicit expression for  $\theta_{N+2}^{(0)}$  in terms of  $f$  and other auxiliary functions.

As a conclusion of rational solutions of Q2, we give the following theorem.

**Theorem 9.** Suppose that  $f = f_N$  is defined as in (71). Rational solutions of Q2 can be given by

$$w_{N+2} = \frac{u_{N+2}^2}{\delta^2} + \frac{\bar{f} + \delta \underline{f}}{\bar{f} - \delta \underline{f}} \left( \frac{-\bar{\underline{f}}^2}{\delta^2 f^2} + \frac{2\bar{\underline{f}}^2 \underline{f} + 2f^2 \bar{\underline{f}}}{\delta f^2 \underline{f}} + \theta_{N+2}^{(0)} - \frac{2\delta f^2 \bar{\underline{f}} + 2\delta f^2 \underline{f}}{f^2 \underline{f}} + \frac{\delta^2 f^2}{f^2} \right), \quad (77)$$

where  $u_{N+2}$  is given by (72c) and  $\theta_{N+2}^{(0)}$  is determined by (75c).

## 6 Conclusions

In the paper we have derived rational solutions for the lpmKdV equation and some lattice equations in the ABS list. We make use of lpmKdV-Q1(0) consistent triplet to construct their rational solutions iteratively. This then becomes a starting point and through the route in Fig.1 to generate solutions for other equations. All these rational solutions are related to a unified  $\tau$  function in Casoratian form,  $f(\alpha) = |\widehat{N-1}|$ , which obeys the bilinear superposition formula (65).

There are several interesting points we would like to remark. First, formula (20) reveals an explicit relation between certain solutions of Q1( $\delta$ ) and Q1(0). This formula holds not only for rational solutions but also for solitons. Once we obtain  $v_{N-2}$  and  $v_N$  from (14), formula (20) gives a solution  $u_N$  to Q1( $\delta$ ), and these solutions provide a solution sequence for the chain (22) which is based on BT (16), i.e. (21). The second thing is about bilinear superposition formula (65) or (44). Casoratian  $f$  with  $\psi_i$  (51) as a basic entry is also a solution of bilinear equation

$$(a+b)\widehat{f}\widetilde{f} + (b-a)\widetilde{f}\widehat{f} = 2bf\widehat{f}, \quad (78)$$

as well as its dual version by switching  $(a, \sim)$  and  $(b, \wedge)$ . (78) can be considered as a bilinear form of Hirota's discrete KdV equation (see [22] and §8.4.1 in [1]). It was also derived from the Cauchy matrix approach as a bilinear form that is related to H1 (see §9.4.3 in [1]). It is also well known that (78) can be derived as a reduction of the Hirota-Miwa equation, of which some rational solutions were derived from several different ways and reductions of few cases was already considered [18,19]. Here we can consider (65) as a bilinear superposition formula of (78) for rational solutions. Since (65) holds for all  $N \in \mathbb{Z}$ , it might be possible to connect (65) with some 3D lattice equations. Finally, let us go back to  $x_i$  defined in (18). It is interesting that all the  $\{\alpha_N\}$  can be expressed in terms of  $x_i$ . Besides, making a comparison between  $P_N$  in (43) and  $f_N$  in (56), they are formally same but with different  $\gamma_i$ : in  $P_N$  all  $\gamma_i$  are arbitrary while in  $f_N$   $\gamma_i \equiv l_0$ . Recalling Lemma 4 in which  $v_N$  can be positive in the first quadrant  $\{n \geq 0, m \geq 0\}$  by taking  $v_N(0,0) > 0$  which can be done by suitably choosing  $\gamma_{2N-1}$  (see (17) as examples). However,  $f$  does not have such a nonzero property. How can we generate arbitrary  $\gamma_i$  in  $f$  (or  $\alpha_N$ ) and investigate nonzero domain of  $f$ ? These questions are left open for further consideration.

## Acknowledgments

This project is supported by the NSF of China (Nos.11371241 and 11631007).

## A Proof of Theorem 5 for H1

Here we prove Theorem 5 which gives Casoratian form of rational solutions of H1.

First, we prove (50a). Noticing that  $\psi_i$  defined in (51) satisfies shift relation

$$\psi_i(l) - a\psi_i(l+1) = (1-a)\widetilde{\psi}_i(l) \quad (79)$$

and  $\psi_i$  (with  $\varrho_i^\pm = \pm\frac{1}{2}$ ) and  $\alpha_j$  defined by (54) actually obey the relation

$$\psi_i(l) = \sum_{j=0}^{\infty} \alpha_j(l) s_i^{2j+1} \quad (80)$$

we have

$$\alpha_i(l) - a\alpha_i(l+1) = (1-a)\alpha_i(l). \quad (81)$$

With such a shift relation and using the technique in [11], for the Casoratians in (55) we find

$$(1-a)^{N-1}\widetilde{f} = |\widehat{N-2}, \alpha(N-1)|, \quad (82a)$$

$$-a(1-a)^{N-1}\widetilde{f} = |\widehat{N-1}, \alpha(N-1)|, \quad (82b)$$

$$-a(1-a)^{N-1}\widetilde{g} = |\widehat{N-2}, N, \alpha(N-1)| + (1-a)^{N-1}(1+aN)\widetilde{f}, \quad (82c)$$

where we have neglected subscript “ $R$ ” without making any confusion.

Substituting (55) and (82) into the downtilde-shifted (50a), for the l.h.s. we reach

$$|\widehat{N}||\widehat{N-2}, \alpha(N-1)| - |\widehat{N-1}, \alpha(N-1)||\widehat{N-2}, N| + |\widehat{N-1}||\widehat{N-2}, N, \alpha(N-1)|, \quad (83)$$

which is zero in light of Lemma 1. In fact, we can replace the  $N$ -th order vector  $\alpha$  with  $(N+1)$ -th order one, introduce an auxiliary  $(N+1)$ -th order column vector  $e_{N+1} = (0, 0, \dots, 0, 1)^T$ , and rewrite

$$f = |\widehat{N-1}, e_{N+1}|, \quad g = |\widehat{N-2}, N, e_{N+1}|, \quad |\widehat{N-2}, \alpha(N-1)| = |\widehat{N-2}, \alpha(N-1), e_{N+1}|;$$

then after taking  $\mathbf{B} = (\widehat{N-2})$ ,  $\mathbf{a} = \alpha(N-1)$ ,  $\mathbf{b} = e_{N+1}$ ,  $\mathbf{c} = \alpha(N)$ ,  $\mathbf{d} = \alpha(N-1)$ , (83) vanishes due to Lemma 1.

Next, to prove (50b) we consider Casoratians  $f$  and  $g$  composed by  $\phi(l) = (\phi_1, \phi_2, \dots, \phi_N)^T$  where

$$\phi_i(n, m, l) = \frac{1}{2}(1+s_i)^{l+l_0}(1-as_i)^{-n}(1+bs_i)^m - \frac{1}{2}(1-s_i)^{l+l_0}(1+as_i)^{-n}(1-bs_i)^m, \quad (84)$$

which satisfies

$$\phi_i(l) + a\widetilde{\phi}_i(l+1) = (1+a)\widetilde{\phi}_i(l). \quad (85)$$

Introduce vector

$$\omega(l) = (\omega_1(l), \omega_2(l), \dots, \omega_N(l))^T, \quad \omega_j = \frac{1}{(2j+1)!} \partial_{s_i}^{2j+1} \phi_i|_{s_i=0}. \quad (86)$$

Noticing the expression (80) for  $\alpha_j(l)$  and relation

$$\phi_i = \frac{1}{(1-a^2s_i^2)^n} \psi_i$$

where we have taken  $\varrho_i^\pm = \pm 1/2$ , we find

$$\omega = A\alpha, \quad (87)$$

where  $A = (a_{ij})_{N \times N}$  is a lower triangular Toeplitz matrix defined by

$$a_{ij} = \begin{cases} 0, & i < j, \\ \frac{\partial_{s_i}^{2(i-j)}}{[2(i-j)]!} \frac{1}{(1-a^2s_i^2)^n} \Big|_{s_i=0}, & i \geq j. \end{cases}$$

Noticing the relation (87) and  $|A| = 1$ , we have

$$f(\omega(l)) = |A|f(\alpha(l)) = f(\alpha(l)), \quad g(\omega(l)) = g(\alpha(l)).$$

Besides,  $\omega_i$  obeys the same shift relation as (85), i.e.

$$\omega_i(l) + a\tilde{\omega}_i(l+1) = (1+a)\tilde{\omega}_i(l), \quad (88)$$

which leads to

$$\begin{aligned} (1+a)^{N-1}\tilde{f}(\omega(l)) &= |\widehat{N-2}, \tilde{\omega}(N-1)|, \\ a(1+a)^{N-1}\tilde{\tilde{f}}(\omega(l)) &= |\widehat{N-1}, \tilde{\omega}(N-1)|, \\ a(1+a)^{N-1}\tilde{\tilde{g}}(\omega(l)) &= |\widehat{N-2}, N, \tilde{\omega}(N-1)| - (1+a)^{N-1}(Na-1)\tilde{\tilde{f}}. \end{aligned}$$

Then one can find the l.h.s. of (50b) yields

$$|\widehat{N}||\widehat{N-2}, \tilde{\omega}(N-1)| - |\widehat{N-1}, \tilde{\omega}(N-1)||\widehat{N-2}, N| + |\widehat{N-1}||\widehat{N-2}, N, \tilde{\omega}(N-1)|,$$

which vanishes as (83).

(50c) and (50d) can be proved similarly.

## B Proof of Theorem 7 for H2

To prove Theorem 7, we rewrite

$$h = s + t, \quad s = |\widehat{N-2}, N+1|_{\mathbb{R}}, \quad t = |\widehat{N-3}, N-1, N|_{\mathbb{R}}.$$

With the relation (81) and using the technique in [11], for the Casoratians (69) we have,

$$\begin{aligned} a(1-a)^{N-2} \left[ \underline{s} + (a^{-1}-1)(\underline{g} + N\underline{f}) \right] &= -|\widehat{N-3}, N, \underline{\alpha}(N-2)|, \\ a(1-a)^{N-2} \left[ \underline{g} + (a^{-1}+N-1)\underline{f} \right] &= -|\widehat{N-3}, N-1, \underline{\alpha}(N-2)|, \\ a(1-a)^{N-2} \underline{f} &= -|\widehat{N-3}, N-2, \underline{\alpha}(N-2)|. \end{aligned}$$

Again, here and after we drop off subscript “ $R$ ” without making any confusion. Then we find that

$$\begin{aligned} &a(1-a)^{N-2} \left\{ f \left[ \underline{s} + (a^{-1}-1)(\underline{g} + N\underline{f}) \right] - (g + Nf) \left[ \underline{g} + (N+a^{-1}-1)\underline{f} \right] + \underline{f}t \right\} \\ &= -|\widehat{N-1}||\widehat{N-3}, N, \underline{\alpha}(N-2)| + |\widehat{N-2}, N||\widehat{N-3}, N-1, \underline{\alpha}(N-2)| \\ &\quad - |\widehat{N-3}, N-2, \underline{\alpha}(N-2)||\widehat{N-3}, N-1, N| \\ &= 0. \end{aligned} \quad (89)$$

Since

$$f(\omega(l)) = f(\alpha(l)), \quad g(\omega(l)) = g(\alpha(l)), \quad s(\omega(l)) = s(\alpha(l)), \quad t(\omega(l)) = t(\alpha(l)),$$

using (88) one has

$$a(1+a)^{N-2}[\tilde{s}(\omega) - (a^{-1}+1)(\tilde{g}(\omega) + N\tilde{f}(\omega))] = |\widehat{N-3}, N, \tilde{\omega}(N-2)|,$$

$$a(1+a)^{N-2}[\tilde{g}(\omega) - (a^{-1} - N + 1)\tilde{f}(\omega)] = |\widehat{N-3}, N-1, \tilde{\omega}(N-2)|,$$

$$a(1+a)^{N-2}\tilde{f}(\omega) = |\widehat{N-2}, \tilde{\omega}(N-2)|.$$

Consequently it reaches

$$\begin{aligned} & a(1+a)^{N-2} \left\{ f \left[ \tilde{s} - (a^{-1} + 1)(\tilde{g} + N\tilde{f}) \right] - (g + Nf) \left[ \tilde{g} - (a^{-1} - N + 1)\tilde{f} \right] + \tilde{f}t \right\} \\ &= |\widehat{N-1}| |\widehat{N-3}, N, \tilde{\omega}(N-2)| - |\widehat{N-2}, N| |\widehat{N-3}, N-1, \tilde{\omega}(N-2)| \\ & \quad + |\widehat{N-2}, \tilde{\omega}(N-2)| |\widehat{N-3}, N-1, N| \\ &= 0. \end{aligned} \tag{90}$$

Then, adding (90) and the uptilde-shifted (89) yields (68a). The other equation in (68) can be proved similarly.

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